$\mathbb{Z}_2 \wr \mathbb{Z}^2$ has $\mathrm{asdim}_B < \infty$

joint WIP with Qingyuan Chen, Alon Dogon, and Brandon Seward

<u>Def</u>: X std Borel space, $E \subset X \times X$ a cBER.
It's HF if $E = \bigcup_{i=1}^{\infty} E_i$ $\quad E_1 \subset E_2 \subset E_3 \subset \dots$

<u>Q. of Weiss</u>: If G is a cfbl amenable gp.
$G \curvearrowright X$. Is OE relation E_G is HF?

Conley, Jackson, Marks, Seward, and Tucker-Drob:

<u>Def</u>: $G \curvearrowright X$ free. $U \subseteq X$, $A \subset\subset G$
$\mathcal{F}_A(U) \subseteq U \times U$ is an CBER on U
generated by (x, ax), $a \in A$.

$\mathrm{asdim}_B(G \curvearrowright X) \leq d$ if $\forall A \subset\subset G$
$\exists$ Borel cover $\{U_0, \dots, U_d\}$ of X s.t.

$\mathcal{F}_A(U_i)$ is finite and, in fact, all the classe are uniformly bdd., that is $\exists B \subset\subset G$ s.t. $x \in U_i$ $[x)_{\mathcal{F}_A(U_i)} \subset Bx$.

Thm: If $\operatorname{asdim}_B(G \curvearrowright X) < \infty \Rightarrow E_c$ is HF.

Idea to get examples: We have some "easy" gps.

$$\{1\} = G_n \lhd G_{n-1} \lhd \dots \lhd G_0 = G$$

G_{i-1}/G_i is "easy".

Def: $\Phi \subset\subset \operatorname{Aut}(H)$

H has Φ-bdd packing

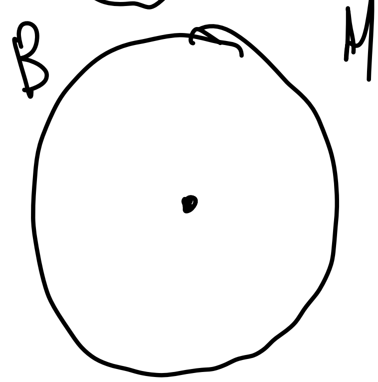
$\exists k \in \mathbb{N}$ s.t. $\forall B_0 \subset\subset H$

$\exists B_0 \subset B \subset\subset H$ s.t. $\Phi(B^3) = \bigcup_{\varphi \in \Phi} \varphi(B^3)$

if $T \subset B^3 \Phi(B^3)$

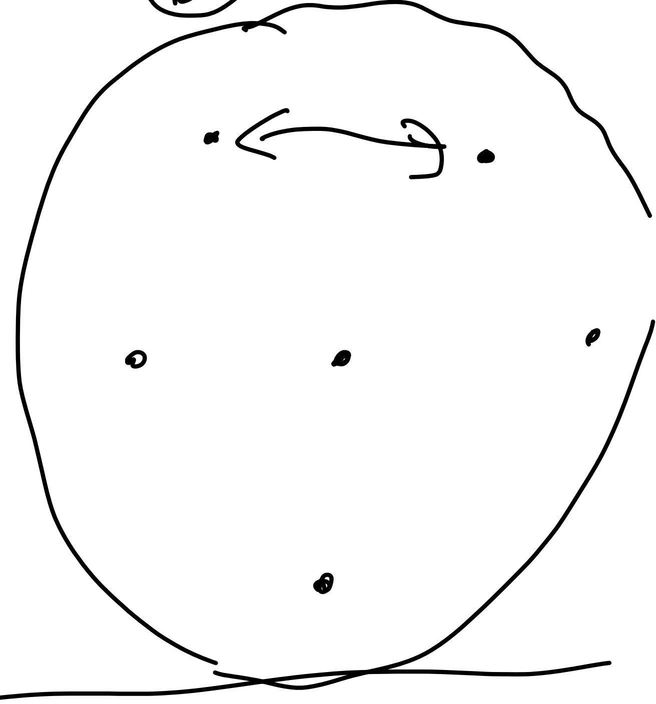
with $(T^{-1}T \setminus \{1\}) \cap B = \emptyset$

then $|T| < k$



B^3

then $|T| \le k$



Examples:

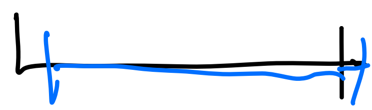
1) $M = \mathbb{Z}^d$, Φ any.

Pf: $\varphi \in Aut(\mathbb{Z}^d) \to M_\varphi \in SL_d(\mathbb{Z})$

$B = B_r(0)$

$B^3 = B_{3r}(0)$ $\quad B^3\Phi(B^3) \subset B_{3r+3r\cdot\max_{\varphi\in\Phi}\|M_\varphi\|}(0)$

2) $\mathbb{Z}_2 \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}$



$1_{\mathbb{Z}} \in \mathbb{Z} \rightsquigarrow \alpha_1 \in Aut\left(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}\right)$

$\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is α_1-bdd. packing

Pf: $B_0 \subset\subset \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ Choose n big enough s.t. $B_0 \subset \bigoplus_{j=-n}^{n} \mathbb{Z}/2\mathbb{Z} =: B$

$B^3 = B$

$\alpha_1(B^3) = \bigoplus_{-n+1}^{n+1} \mathbb{Z}/2\mathbb{Z}$

$B^3\alpha_1(B^3) = \bigoplus_{-n}^{n+1} \mathbb{Z}/2\mathbb{Z}$

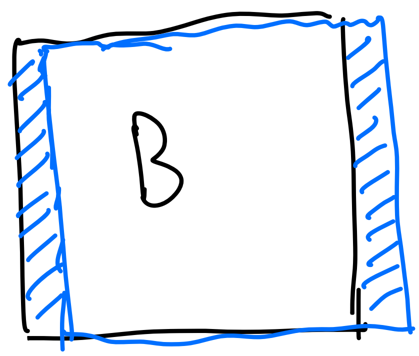
$[B^3\alpha_1(B^3) : B] = 2$ ✓

3) finite grps.

→ 4) torsion-free abelian gps with finite $\mathbb{Q}$-rank, φ any.

Non-example: $\mathbb{Z}_2 \wr \mathbb{Z}^2$

$\mathbb{Z}^2$ $\quad \alpha_{(1,0)}, \alpha_{(0,1)} \subset Aut\left(\bigoplus_{\mathbb{Z}^2} \mathbb{Z}_2\right)$



Def: $H \triangleleft G$. (H,G) has good asymptotics if $\forall$ free action $G \curvearrowright X$

1) $\operatorname{asdim}_B(H \curvearrowright X) < \infty$

2) $\forall B \subset\subset G \ \exists \ell \in \mathbb{N} : \forall A \subset\subset H$

$$\chi_B(S_{BA\setminus H}) \le \ell,$$

where $S_{BA\setminus H} \subset X \times X$

$$S_{BA\setminus H} = \{(x,y) \mid x \in (BA\setminus H)y \text{ or } y \in (BA\setminus H)x\}$$

Note that $(\{1\}, G)$ has GA $\mid G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_n = \{1\}$

Lemma:

If (G_i, G) has GA and $\underbrace{G_{i-1}/G_i}$ has $\underbrace{G/G_i}$- bdd packing $\Rightarrow$

$\Rightarrow (G_{i-1}, G)$ has GA.

Idea: Try to prove GA directly.

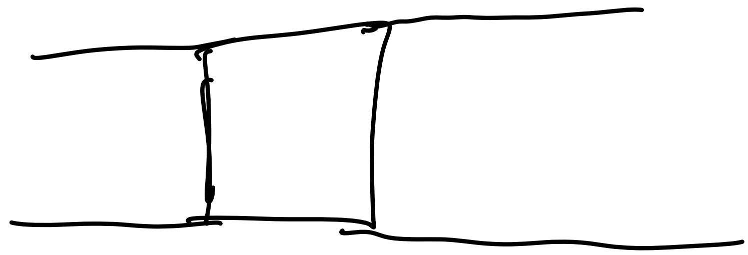
Proposition: $\left(\bigoplus_{\mathbb{Z}^2} \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}_2 \wr \mathbb{Z}^2\right)$ has GA.

More generally, any loc. fin. abelian gp. has GA in any larger group.

$\rightarrow G = G_0 \rhd G_1 \rhd \cdots \rhd G_{n-1} \rhd G_n = \{1\}$

G_{n-1} loc. fin. abelian,

$\chi\left(\mathcal{G}_{(1,0)} A\right) = 2$



$\langle A, (1,0) \rangle$